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MINIMAX CONTROL OF A SYSTEM WITH PROCESS UNCERTAINTY IN SUBSTATE SPACE

by

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ABSTRACT

The control of a single input-single output plant with parameter uncertainty has been approached using a minimax technique. A linear time invariant controller results which requires only partial state feedback and is optimal in the sense that it minimizes a quadratic criterion involving tracking error, control, and parameter uncertainty. Bounded input-bounded output stability is guaranteed provided the transfer function has only left half plane zeros. If uncertainties are bounded, it is always possible to stabilize the system when sufficient control amplitude is available. The number of states required to generate the control is equal to the system order less the number of zeros.

I. INTRODUCTION

Control of a dynamic single input-single output system having parameter uncertainty and unknown process nonlinearity has been considered using a minimax technique. Specifically, we intend to show that a controller can be designed using a nominal model to insure a satisfactory performance of the system inspite of ignorance of system parameters. The problem is posed with the additional constraint that the controller is linear and that it requires only partial state feedback.

Lyapunov type synthesis techniques [1,2,3] may be applied to control this class of plants. The resulting controller is highly nonlinear. The limitation of this approach is not only the complexity of the controller structure, but also the lack of any insight as how to determine the control signal amplitude, It is sometimes possible to obtain a design for this class of plant using Stochastic Control Theory. The general problem[4] here is to determine the least favourable distribution for the uncertain parameter vector $\underline{\xi}$. Another alternate approach is to use minimax control rule[5].

In this paper, the problem has been approached by minimizing and maximizing a quadratic performance index involving the tracking error, the control, and the "uncertainty signal". The resulting controller is linear and is optimal in the sense that it minimizes the above performance criteria. The number of states required to generate the control is equal to the system order less the number of zeros. Bounded input-bounded output stability is guaranteed provided the transfer function is of minimum-phase type. If the uncertainties are bounded, it is always possible to stabilize the system, irrespective of the location of open loop poles of the system, if sufficient control is available. These results also apply for systems with rather general nonlinearities that do not involve the control. It has also been shown that the tracking error

vector admits an upper bound and that the bound can be made arbitrarily small.

II. MOTIVATION OF THE PROBLEM

We consider the single-input-single output system described as in (1)-(3):

$$\frac{X_1(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} = \frac{N(s)}{D(s)} \qquad m<n \tag{1}$$

where $X_1(s)$ and $U(s)$ represents the Laplace transforms of output $x_1(t)$ and input $u(t)$ respectively. The equation (1) may be reproduced in the state variable form[6]

$$\underline{\dot{x}} = A\underline{x} + \underline{h}u = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_{n-m} \\ \vdots \\ h_n \end{bmatrix} u \tag{2}$$

where $h_i = \begin{cases} 0 & i<n-m \\ b_{n-k} - \sum\limits_{i=n-m}^{k-1} a_{n-k+i}h_i & i \leq n-m \end{cases}$.

Since $x_{i+1} = \dot{x}_i$, $i=1,2,\dots, n-m$, it can be seen from (1) that

$$\frac{X(s)}{U(s)} = (sI-A)^{-1}\underline{h} = \frac{1}{D(s)} \begin{bmatrix} N(s) \\ sN(s) \\ \vdots \\ s^{n-m-1}N(s) \\ N_{n-m}(s) \\ \vdots \\ N_n(s) \end{bmatrix} \tag{3}$$

where $N_{n-k}(s)$, $k=0,1,\ldots,m$, are polynomials in s. Consider now the use of a linear feedback law

$$u = -k_0\,[k_1x_1 + k_2x_2 + \ldots\ldots + k_{n-m}x_{n-m}] = -k_0\underline{k}^T\underline{x} \tag{4}$$

where $\underline{k}^T = [k_1, k_2, \ldots\ldots, k_{n-m}, 0, \ldots .0]$ is a constant vector.

The eigenvalues of the closed loop system are then solutions of

$$0 = |sI - A + k_0\underline{hk}^T| = |sI-A\,||\,I + k_0\,(sI-A)^{-1}\underline{hk}^T|$$

$$= |sI-A|\;(1 + k_0\underline{k}^T\,(sI-A)^{-1}\underline{h}) \tag{5}$$

where $|A|$ is the determinate of matrix A.

The last equality of (5) is obtained using the identity

$$|I + \underline{C}\,\underline{D}\,| = 1 + \underline{D}^T\underline{C} \tag{6}$$

$\underline{C}$, $\underline{D}$ being vectors of compatible dimensions. Furthermore, combining (4) and the definition of $\underline{k}$ with (5) yields

$$|sI-A + k_0\underline{hk}^T| = D(s)\;[\,1 + k_0\frac{N(s)(k_1 + k_2s + \ldots\ldots + k_{n-m}s^{n-m-1})}{D(s)}]$$

$$= D(s)\;[1 + k_0\,\frac{N(s)\;k(s)}{D(s)}\,] = 0 \tag{7}$$

(7) can also be obtained using Figure 1 which illustrates the system with feedback. The characteristic equation of the closed loop system is

$$1 - \text{Loop Gain} = 1 + k_0\underline{k}^T\,(sI-A)^{-1}\underline{h} = 0,$$

which agrees with (5) and (7). We know as $k_0 \to \infty$, that zeros of (7) approach the n-1 finite zeros of $N(s)\cdot k(s)$ and one zero at $-\infty$. Hence if $N(s)$, $k(s)$ are Hurwitz polynomials with b_m, $k_{n-m} > 0$, the system is stable for k_0 sufficiently large - regardless of the zeros of $D(s) = |sI-A|$. Furthermore, response to any bounded input $R(s)$ will be bounded. The problem is to choose the nonzero

elements of $\underline{k}$ so that in addition to stability the system exhibits behavior which is in some sense good. Furthermore we should like a design procedure which can yield this good behavior despite uncertainties in the system parameters.

III. SYSTEM DESCRIPTION, DEVELOPMENT AND PROBLEM FORMULATION

Define a stable model

$$\dot{\underline{y}} = A_0\underline{y} + \underline{\beta}_0 r = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-m-1} \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta_0 \end{bmatrix} r \qquad (8)$$

and a system

$$x_1^{(n)} + a_{n-1}x_1^{(n-1)} + \ldots + a_0 x_1 = b_m u^m + b_{m-1}u^{m-1} + \ldots + b_1\dot{u} + b_0 u + f(x_1, \ldots, x_{n-m}, t) \qquad (9)$$

where $\underline{y}$ and r are the model output and reference input respectively, and $f(x_1, x_2 \ldots x_{n-m}, t)$ is a nonlinear function.

If

(1) $b_m > \beta > 0$ and $b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0$ Hurwitz, and if

(2) $f(x_1, x_2 \ldots, x_{n-m}, t)$ is a bounded continuous nonlinear function,

we wish to show that the error

$$\underline{e} = (x_1, x_2 \ldots, x_{n-m})^T - \underline{y} \triangleq \underline{x}^* - \underline{y} \qquad (10)$$

can be bounded with an arbitrarily small bound, despite imperfect knowledge of a_i, b_k, $i = 0, 1, \ldots, n$, $k = 0, 1, \ldots m$. This will be achieved by a linear feedback law which requires only partial state feedback and is in a sense optimal. Furthermore u will be similar in form to (4) with k(s) Hurwitz.

Remark 1.

a) n,m, are integers which may be unknown but the difference (n-m) is assumed to be known. Also a lower bound of the highest order derivative coefficient of u is available.

b) Note that (9) is identical to the linear system (1) except for addition of the nonlinear term.

c) When a regulator is being designed it is admissable to let $r=\underline{y}=0$.

We proceed by rewriting (9)

$$x_1^{(n)} + \alpha_{n-m-1} x_1^{(n-1)} + \ldots + \alpha_0 x_1^{(m)} - \beta u^m = \sum_{j=0}^{m-1} b_j u^{(j)} + (b_m - \beta) u^m + \sum_{j=m}^{n-1} \alpha_{j-m} x_1^{(j)} - \sum_{j=0}^{n-1} a_j x_1^{(j)} + f(x_1, \ldots, x_{n-m} t) \tag{11}$$

Integrating each side m times, gives

$$x_1^{(n-m)} + \alpha_{n-m-1} x_1^{(n-m-1)} + \ldots + \alpha_0 x_1 = \beta u + \xi(t) \tag{12}$$

where $\xi(t)$ is the m fold integral of the right side of (11) together with initial condition terms. In state variable form,(12) may be written

$$\dot{\underline{\overset{*}{x}}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{n-m-1} & \cdot & \cdot & \cdots & -\alpha_0 \end{bmatrix} \underline{\overset{*}{x}} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \xi(t) \end{bmatrix}$$

$$= A_0 \underline{\overset{*}{x}} + \underline{\beta} u + \underline{\xi}(t), \tag{13}$$

where $\underline{\overset{*}{x}}$ corresponds to the first n-m elements of $\underline{x}$ as in (10)

i.e. $\underline{\overset{*}{x}} = [x_1, \dot{x}_1, \ldots, x_1^{(n-m-1)}]^T = [x_1, x_2, \ldots, x_{n-m}]^T.$ (14)

Consequently subtracting (8) from (13) yields

$$\dot{\underline{e}} = (\dot{\overset{*}{\underline{x}}} - \dot{\underline{y}}) = A_0 e + \underline{\beta}\,(\tilde{u} + \tilde{\xi}),\ \tilde{u} = (u - \frac{\beta_\theta}{\beta} r), \tilde{\xi} = \frac{1}{\beta}\xi. \tag{15}$$

$\tilde{\xi}(t)$, of course, generates differences between the model and $\overset{*}{\underline{x}}$. If $\tilde{\xi} = 0 = \tilde{u}$, $\underline{e}(t)$ approaches zero asymptotically since A_0 has negative eigenvalues. Furthermore, even if $\tilde{u} = 0$, $\tilde{\xi} \neq 0$, $\underline{e}$ will be bounded for $\tilde{\xi}(t)$ bounded. The problem is to realize a $\tilde{u}$ which will not only retain stability but allow a bound on $\underline{e}$ to be made arbitrarily small.

IV. CONTROLLER DESIGN

For the purpose of controller design consider the criterion

$$J = \frac{1}{2}\int_0^\infty (\underline{e}^T Q \underline{e} + \tilde{u}^2 R - \tilde{\xi}^2 L)\, dt \tag{16}$$

with R, L > 0 and Q positive definite. $\tilde{u}$, $\tilde{\xi}$ will be chosen to minimize and maximize J respectively. It is readily shown that the optimum [7] $\overset{*}{\tilde{u}}$, $\overset{*}{\tilde{\xi}}$ are given by

$$\overset{*}{\tilde{u}} = -\frac{1}{R}\underline{\beta}^T P \underline{e},\quad \overset{*}{\tilde{\xi}} = \frac{1}{L}\underline{\beta}^T P \underline{e} \tag{17}$$

where P is the symmetric matrix satisfying

$$PA_0 + A_0^T P + Q - P\underline{\beta}\underline{\beta}^T P\,(\frac{1}{R} - \frac{1}{L}) = 0\ . \tag{18}$$

Furthermore, (i) $A_0 - \frac{1}{R}\beta\beta^T P$ has negative eigenvalues

(ii) P is the unique positive definite steady state solution of $-\dot{P} = PA_0 + A_0^T P + Q - P\beta\beta^T P\,(\frac{1}{R} - \frac{1}{L})$, $R \leq L$.

(iii) $J[\underline{e}_0; \overset{*}{\tilde{u}}, \overset{*}{\tilde{\xi}}\,] = \min_{\tilde{u}} \max_{\tilde{\xi}} J[\underline{e}_0; \tilde{u}, \tilde{\xi}\,] = \max_{\tilde{\xi}} \min_{\tilde{u}} J[\underline{e}_0; \tilde{u}, \tilde{\xi}\,]$

$$= \frac{1}{2}\underline{e}_0^T P\, \underline{e}_0$$

(17) will, in one sense, yield a conservative design since $\tilde{\xi}$ is assumed to act in the most perverse manner. In another sense, however, $\tilde{\xi}$ need not abide by the rules of the game and may be using a smaller L than "agreed upon". This may require that R in turn be decreased.

Remark 2.

It should be noted that a unique positive definite solution P of (18) exists [8] if $R \leq L$. In case, $R<L$ the implication is that we are trying to balance the effect of the "uncertainty signal" with a larger amplitude of control signal than what is needed with $R = L$. Consequently for much of the rest of the paper, we shall restrict ourselves to the case $R = L$. In case $R<L$, it is still possible to show that proposition 1 (mentioned below) is true. In case $R>L$, nonuniqueness of solution P of (18) poses some problems in the subsequence analysis of the error bound and will be discussed in a future paper.

For convenience, define $\underline{k}$ as the last row or column of P. Then

$P\underline{\beta}^T = [\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_{n-m}][0,0,\ldots,\beta]^T = \beta\underline{p}_{n-m} \triangleq \beta\underline{k}$.

from (17), $\overset{*}{u}$ is given by

$$\overset{*}{u} = -\frac{1}{R}\beta\underline{p}_{n-m}^T\,\underline{e} = -\frac{\beta}{R}\underline{k}^T\underline{e}\,. \tag{19a}$$

Accordingly

$$\overset{*}{U}(s) = -\frac{\beta}{R}\underline{k}^T[1,s,\ldots,s^{n-m-1}]^T\,E(s) = -\frac{\beta}{R}k(s)E(s) \tag{19b}$$

where

$$k(s) = \underline{p}_{n-m}^T\,[1,s,s^2\ldots,s^{n-m-1}]^T\,.$$

Now it will be helpful, at this stage, to establish an important property of this control signal $\tilde{u}$ that is outlined in the following proposition:

<u>Proposition 1</u>

With A_0 stable and as defined in (13), k(s) is a stable polynomial for $R=L$.

<u>Proof:</u>

Consider the system defined by (15) and represented in Figure 2. The open loop transfer function, z(s), between $\xi(s)$ and u(s), is given by

$$z(s) = \left[\frac{\beta^T}{R}P\,(sI-A_0)^{-1}\beta\right] = \beta^2\,\frac{k(s)}{D_1(s)} \tag{20}$$

where $D_1(s) \triangleq |sI-A_0|$ is a stable polynomial and $k(s)$ is as defined in (19b). Now using (18), we obtain

$$2\mathrm{Re}[z(s)]=z(\overset{*}{s}) + z(s) = \frac{1}{R}\,[\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,P\underline{\beta} + \underline{\beta}^T P\,(sI-A_0)^{-1}\beta]$$

$$= \frac{1}{R}\,\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,[P(s+\overset{*}{s}) + P\underline{\beta}^T\,(R^{-1}-L^{-1})\,\underline{\beta}P + Q]\,(sI-A_0)^{-1}\beta$$

$$= 2\{\mathrm{Re}s\}\,[\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,P(sI-A_0)^{-1}\underline{\beta}] + \underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,Q(sI-A)^{-1}\underline{\beta}$$

for $R = L$.

Here $\overset{*}{s}$ denotes the complex conjugate of s. Since P, Q are positive definite, $\mathrm{Re}[z(s)]$ is ~~nonnegative~~ for $\mathrm{Re}(s)>0$. Therefore the transfer function $z(s)$ is positive real[9] which implies from (20) that $k(s)$ is Hurwitz.

V. DERIVATION OF A BOUND ON THE ERROR

In order to determine a bound on $\underline{e}$ when $\tilde{u}$ satisfies (17) but $\tilde{\xi}(t)$ is arbitrary, let

$$V(e) = \frac{1}{2}\underline{e}^T P\underline{e} \tag{21}$$

where P is the positive definite matrix satisfying (18). $\underline{e}$ must now satisfy the differential equation (15) with $\tilde{u}$ as in (19a); that is

$$\underline{\dot{e}} = [A_0 - \frac{\beta}{R}\,\beta^T P]\,\underline{e} + \underline{\beta}\,\tilde{\xi}(t). \tag{22}$$

The time derivative of (21) is

$$\dot{V}(e) = \frac{1}{2}e^T[A_0 - \frac{1}{R}\,\underline{\beta\beta}^T P)^T\,P + P\,(A_0 - \frac{1}{R}\,\underline{\beta}\,\underline{\beta}^T P)e]\,\underline{e} + \tilde{\xi}\,(t)\underline{\beta}^T P\underline{e}$$

$$= -\frac{1}{2}\underline{e}^T Q\underline{e} - \frac{1}{2}\,\beta^2(\underline{k}^T\underline{e})^2(\frac{1}{L} + \frac{1}{R}) + \tilde{\xi}\,(t)\beta\,(\underline{k}^T\underline{e}) \tag{23}$$

where (18), (19a), and (22) have been used to refine the result. Clearly if $|\tilde{\xi}(t)|$ is bounded, (23) will be negative for $||\underline{e}||$ sufficiently large and admit an upper bound $||e||$. First we must examine $\tilde{\xi}(t)$, however, by considering the behavior of the full system described by (9), (10), (21). We shall return to complete our examination of (23).

VI. STABILITY OF THE OVERALL SYSTEM

Turning now to the total system, (9) may be written in state form, basically as in (2),

$$\dot{\underline{x}} = A\underline{x} + \underline{h}u + \underline{f}\ (\overset{*}{\underline{x}},\ t) \tag{25}$$

with $\underline{f}\ (\overset{*}{\underline{x}},t) = [0,\ 0,\ .\ .\ .,\ 0,\ f(\overset{*}{\underline{x}},t)]^T$.

If we now define

$$\underline{x} = [\begin{matrix}\overset{*}{x} \\ \underline{v}\end{matrix}],\ A = \begin{bmatrix} A_* & A_{*v} \\ A_{v*} & A_v \end{bmatrix}$$

$$\underline{Z} = [\begin{matrix} e \\ \underline{v}\end{matrix}]\ ,\ \underline{v} = (v_1,\ v_2,\ .\ .\ .\ v_m)^T \tag{26}$$

and combined (25), (8) and (10), to eliminate $\overset{*}{\underline{x}}$, the result is

$$\overset{*}{\underline{Z}} = A\underline{Z} + \underline{h}u + \begin{bmatrix} A_* & -A_0 \\ & A_{v*} \end{bmatrix} \underline{y} - \begin{bmatrix} \beta_0 \\ 0 \end{bmatrix} r + \underline{f}\ (\underline{e} + \underline{y},\ t)$$

$$= A\underline{Z} + \underline{h}u + \underline{\nu} \tag{27}$$

where

$$\underline{\nu} = [0,0,.\ .\ .,\ 0,\ \underset{\substack{\uparrow \\ (n-m)^{th}\ \text{entry}}}{\sum_{i=0}^{n-m} \alpha_{i-1}y_i - \beta_0 r},\ 0,\ 0,\ .,\ f(\underline{y} + \underline{e},\ t)\ - \sum_{i=1}^{n-m} a_{i-1}y_i]^T. \tag{28}$$

$\underline{\nu}$ is bounded if the model input r, is bounded as assumed. If the "optimal" u given by (19a)is used in (27), the closed loop system satisfies

$$\dot{\underline{Z}} = [A - \frac{\beta}{R}\,\underline{h}\,\underline{\hat{k}}^T]\ \underline{Z} + \underline{h}\,\frac{\beta_0}{\beta}\,r + \underline{\nu}$$

$$= [A - \frac{\beta}{R}\,\underline{h}\,\underline{\hat{k}}^T]\ \underline{Z} + \underline{\nu}' \tag{29}$$

with

$$\underline{\hat{k}}^T = [k_1,\ k_2,\ .\ .\ .\ k_{n-m},\ 0,\ 0,\ .\ .\ .\ 0]$$

$$\underline{\nu}' = \underline{\nu} + \underline{h}\ \frac{\beta_0}{\beta}\ r \triangleq [0,\ 0,\ .\ .\ .\ 0,\ \nu'_{n-m}.\ .\ .\ \nu'_n]. \tag{30}$$

The characteristic equation of the closed loop system is obtained as in (5)-(7), i.e.

$$0 = |sI-A + \frac{\beta}{R} \underline{h}\, \underline{\hat{k}}^T| = |sI-A| \quad |I + (sI-A)^{-1} \frac{\beta}{R} \underline{h}\, \underline{\hat{k}}^T| = |sI-A| \; (1 + \frac{\beta}{R} \hat{k}^T (sI-A)^{-1} \underline{h})$$

$$= D(s) \; [1 + \frac{\beta}{R} \; \frac{N(s)}{D(s)} \; (k_1 + k_2 s + \; . \; . \; . \; . \; + k_{n-m} s^{(n-m-1)}]$$

$$= D(s) + \frac{\beta}{R} N(s) \; k(s) \qquad (31)$$

Now as $R \to 0$, (n-1) roots of (31) approach zeros of N(s)k(s), and the last root goes to infinity along the negative real axis. Since N(s), k(s) are "stable polynomials", then (30) is stable for R adequately small.

Consequently as the penalty R on control is reduced, permitting larger control amplitudes, the system (1) - (2) or (27) is stable for the feedback law (19) provided N(S) is a stable polynomial with $b_m \neq 0$.

Now since (29) yields a bounded $\underline{Z}$ with a bounded input $\underline{\nu}'$, all the elements of $\underline{Z}$ are bounded. In fact, bounds on $\underline{\nu}'$ do not depend on $\underline{Z}$ but are determined mainly on $\underline{y}$. From (29) - (31) it can be seen, in a straightforward manner, that

†

$$\mathcal{L}(e_i) = E_i(s) = \frac{1}{D(s) + \frac{\beta}{R} N(s) k(s)} \quad \sum_{j=n-m}^{n} f_j^i(s) \; \nu_j'(s) \qquad i=1,2,\ldots,(n-m-1)$$

$$\mathcal{L}(v_i) = V_i(s) = \frac{1}{D(s) + \frac{\beta}{R} N(s) \; k(s)} \sum_{j=n-m}^{n} [g_j^i(s) + \frac{1}{R} \ell_j^i(s)] \; \nu_j'(s), \quad i=1,2,\ldots,m,$$

where $f_j^i(s)$, $g_j^i(s)$ and $\ell_j^i(s)$ are polynomials in s, independent of R, and of order $\leq$ (n-1).

As $R \to 0$, $|E_i(s)| \to 0$, $i=1,2,\ldots,(n-m-1)$

and

$$|V_i(s)| \to \left| \frac{\sum_{j=n-m}^{n} \ell_j^i(s) \; \nu_j'(s)}{N(s) k(s)} \right| \qquad i=1,2,\ldots,m.$$ Thus an ultimate

bound on V_1 exists and is independent of R as $R \to 0$. Furthermore the error bound, i.e. the bound on $||\underline{e}||$ goes to zero as $R \to 0$. We explore this further.

VII. FURTHER RESULTS ON THE BOUND OF THE ERROR

Let us now examine $\xi(t)$. From (22)

$$\xi(t) = \beta\tilde{\xi}(t) = \dot{e}_{n-m} + \sum_{i=1}^{n-m} a_{i-1}e_i + \frac{\beta^2}{R}\,\underline{k}^T\underline{e}$$

$$= \dot{e}_{n-m} + \sum_{i=1}^{n-m} (a_{i-1} + \frac{\beta^2}{R}\,k_i)e_i$$

From (29), on the otherhand

$$\dot{z}_{n-m} = \dot{e}_{n-m} = v_1 - \frac{\beta}{R}\,b_m\underline{k}^T\underline{e} + \sum_{i=0}^{n-m} \alpha_{i-1}y_i - \beta_0 r + b_m\frac{\beta_0}{\beta}r$$

Thus

$$\xi(t) = v_1 + \sum_{i=1}^{n-m} [a_{i-1}e_i + \alpha_{i-1}y_i] + \beta_0(\frac{b_m}{\beta} - 1)\,r + \frac{\beta}{R}\,(\beta - b_m)\underline{k}^T\underline{e}$$

$$= \xi_1 + \frac{\beta}{R}\,(\beta - b_m)\underline{k}^T\underline{e} \qquad (32)$$

where

$$\xi_1 = v_1 + \sum_{i=1}^{n-m} [a_{i-1}e_i + \alpha_{i-1}y_i] + \beta_0(\frac{b_m}{\beta} - 1)$$

has an upper bound which is independent of R and exists as $R \to 0$.

Returning to (23) with (32) replacing $\xi(t)$

$$\dot{V}(\underline{e}) = -\frac{1}{2}\,\underline{e}^TQ\underline{e} - (\underline{k}^T\underline{e})^2(\frac{b_m\beta}{R} - \frac{\beta^2}{2R} + \frac{\beta^2}{2L}) + \xi_1\underline{k}^T\underline{e}$$

$$\leq -\frac{1}{2}\,\underline{e}^TQ\underline{e} - \frac{1}{2}\,(\underline{k}^T\underline{e})^2\,\beta^2\,(\frac{1}{R} + \frac{1}{L}) + \xi_1\underline{k}^T\underline{e}$$

$$\leq -\frac{1}{2}\,\underline{e}^TQ\underline{e} - \frac{1}{2}\,(\underline{k}^T\underline{e})^2\,\beta^2(\frac{1}{R} + \frac{1}{L}) + |\xi_1|_{max}|\,\underline{k}^T\underline{e}\,| \qquad (33)$$

$$\leq -\frac{1}{2}\,\underline{e}^TQ\underline{e} + \frac{1}{2}\,\frac{|\xi_1|^2_{max}}{(\frac{1}{L} + \frac{1}{R})\beta^2} = W(\underline{e})$$

The last inequality above is obtained by maximizing the last two previous terms with respect to $|\underline{k}^T\underline{e}|$. It is possible to show in a straight forward manner that W(e) and hence $\dot{V}(e)$ is negative for all

$$||\underline{e}|| = (\underline{e}^T\underline{e})^{1/2} \geq R_a = \frac{|\xi_1|_{max}}{\beta} \sqrt{\frac{1}{(\lambda_Q)_{min}(\frac{1}{L} + \frac{1}{R})}}$$

where $(\lambda_Q)_{min}$ denotes the minimum eigenvalue of Q. This, of course, is conservative since we have permitted $\underline{e}$ to both maximize certain terms in (33) and to maximize $||e||$ subject to $W(e) \geq 0$.

The bound on $||e||$ can now be found in standard fashion;

(1) Determine V_0 = maximum V(e) subject to $||\underline{e}|| \leq R_a$

(2) Determine R_b = maximum $||e||$ subject to $V(e) \leq V_0$

$\underline{e}$ must asymptotically approach the region defined by $V(\underline{e}) \leq V_0$.

R_b represents an ultimate bound on $||e||$ since $\underline{e}$ asymptotically approaches the region defined by $V(e) \leq V_0$ but $\dot{V}(e)$ may be indefinite therein. Thus $||e||$ is bounded by R_b where

$$R_b = R_a \sqrt{\frac{(\lambda_P)_{max}}{(\lambda_P)_{min}}} = |\xi_1|_{max} \sqrt{\frac{(\lambda_P)_{max}}{(\lambda_P)_{min}(\lambda_Q)_{min}\ (\frac{1}{R} + \frac{1}{L})\beta}}$$

as shown in the Figure 4. This bound can be made arbitrarily small by relaxing the penalty R on the control u since $|\xi_1|_{max}$ admits on upper bound as can be seen from previous section and (32).

VII EXAMPLE

To illustrate the preceeding analysis, consider the following open-loop unstable plant described by

$$\dddot{x} + \ddot{x} - \dot{x} + x - \text{Sin}x + x^2 = \ddot{u} + 2\dot{u} + u \qquad (34)$$

with an arbitrary first order model described by

$$\dot{y} + 2y = r$$

where r is a step input. It is to be noted that the minimum order of the model is specified by the difference between the number of poles and zeros of the plant. This is, in this example, one. With little manipulation as shown in the analysis and integrating (34) twice, the error equation can be expressed as

$$\dot{e} + 2e = \xi + \tilde{u} \tag{35}$$

$$\tilde{u} = u-r$$

It is desired to find an optimal control $\overset{*}{\tilde{u}}$ by maximizing w.r.t. ξ and minimizing w.r.t. $\tilde{u}$ the performance criterion

$$J = \int_0^\infty (4e^2 + R\tilde{u}^2 - L\xi^2) \; dt, \; R=L \tag{36}$$

subject to (35).

The resulting control is given by

$$\overset{*}{\tilde{u}} = -R^{-1}e \tag{37}$$

and is applied to the original system (34).

The outputs of the plant and model are shown in Figure (5) for different values of R. It can be seen that the error decreases monotonically with the decrease of penalty R. Also plotted are the remaining states $x_2 = \dot{x}$, $x_3 = \ddot{x}$ for different values of R. These are bounded as can be seen in figure (8) and (9). The control signal characteristics are shown in Figure (7).

IX. CONCLUSION

A linear time invariant controller has been designed for a single input-single output system with parameter uncertainty. The number of states required to generate the control signal is equal to the system order less the number of zeros. This is obtained by minimizing a quadratic performance index involving the tracking error, the control signal and the "uncertainty signal". This however, yields a conservative design since the "uncertainty signal" is assumed to act in the most unfavourable manner. Bounded input-bounded output stability is guaranteed provided the transfer function is of minimum-phase type. If the uncertainties are bounded, it has been shown that the system can always be stabilized if sufficient control amplitude is available. These results also apply for systems with rather general nonlinearities that do not involve the control. We have also shown that the tracking error admits an upper bound and that the bound can be made arbitrarily small. Extension of this result to multivariable systems is not immediate and is currently under investigation.

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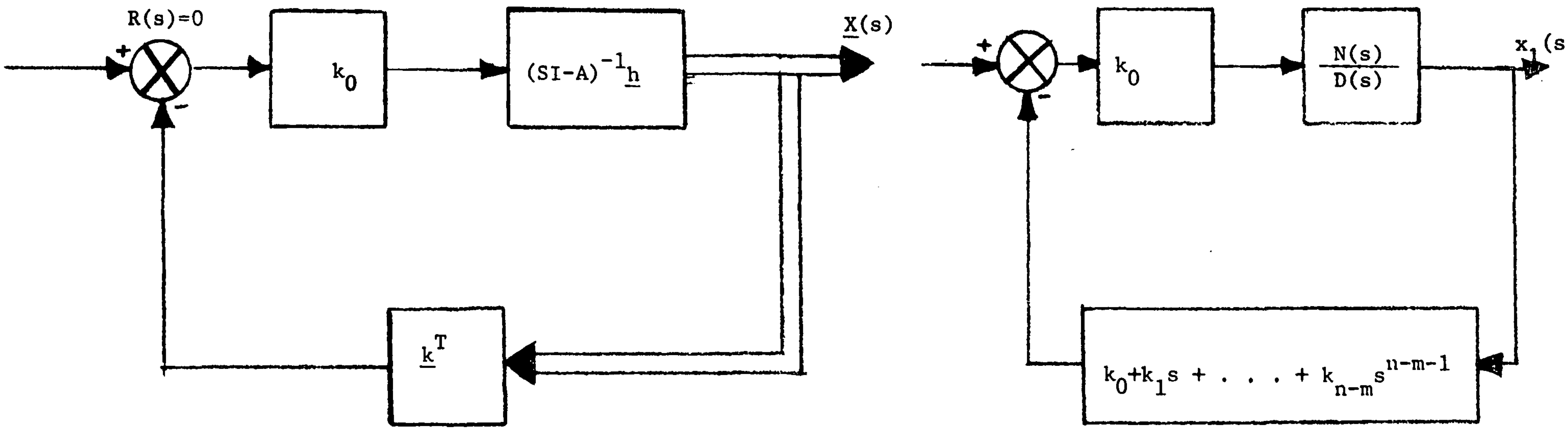


FIGURE 1 A SYSTEM WITH INCOMPLETE FEEDBACK

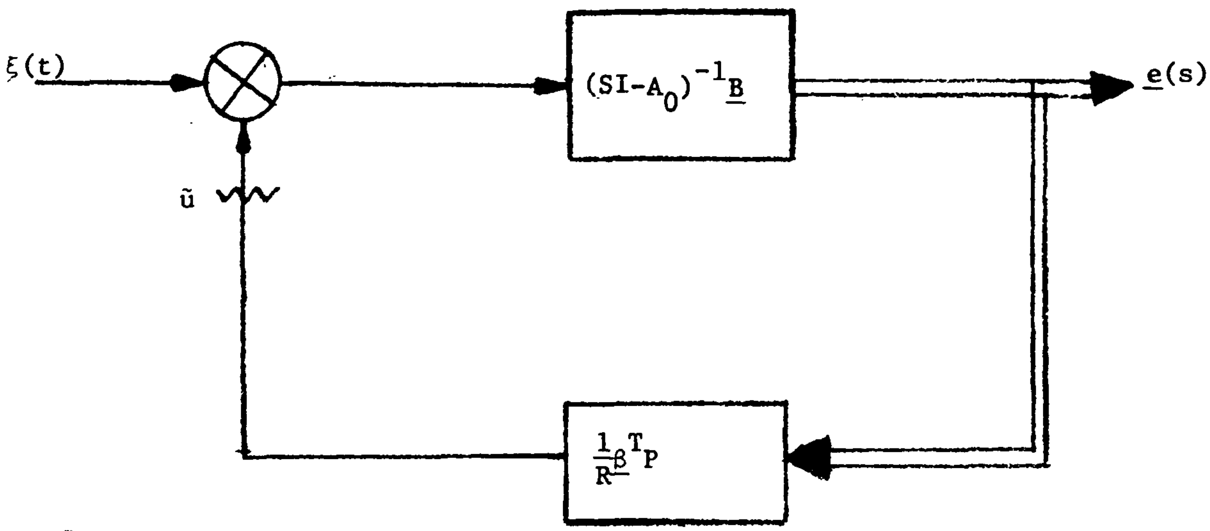


FIGURE 2

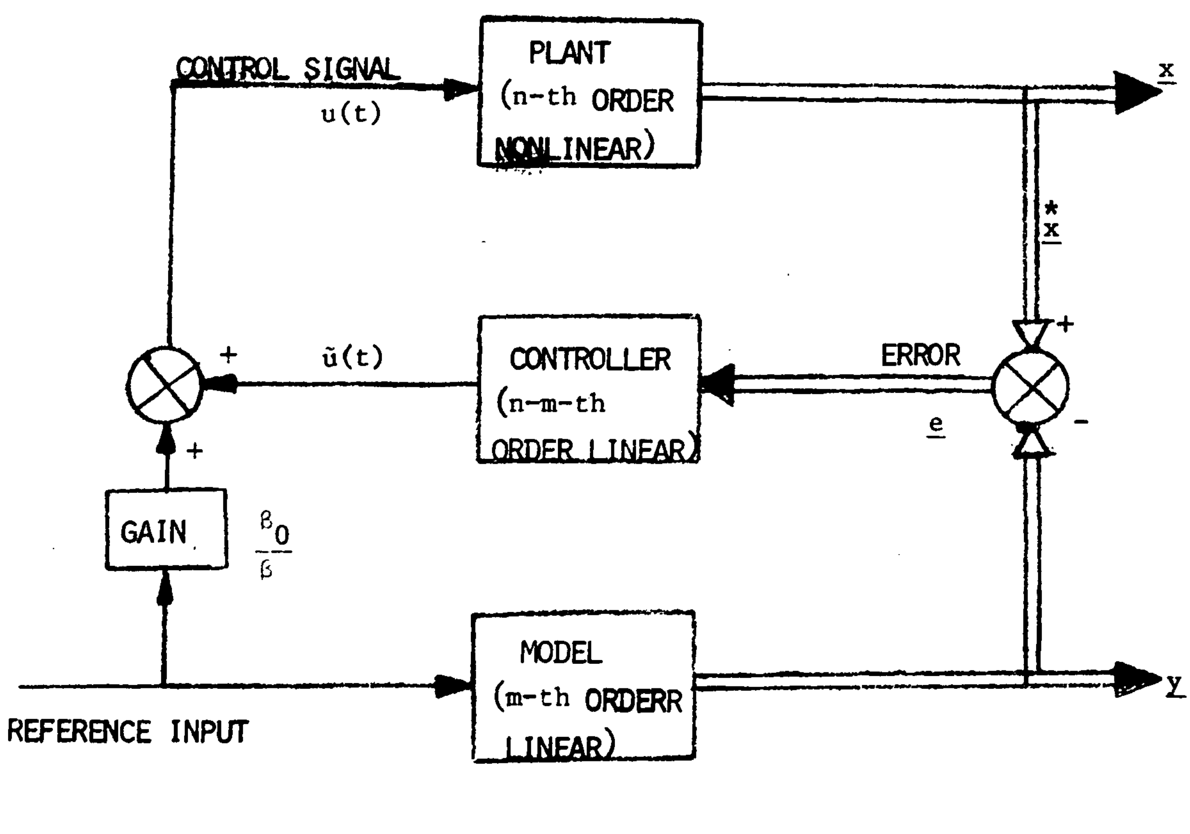


FIGURE 3 - MODEL REFERENCE SYSTEM

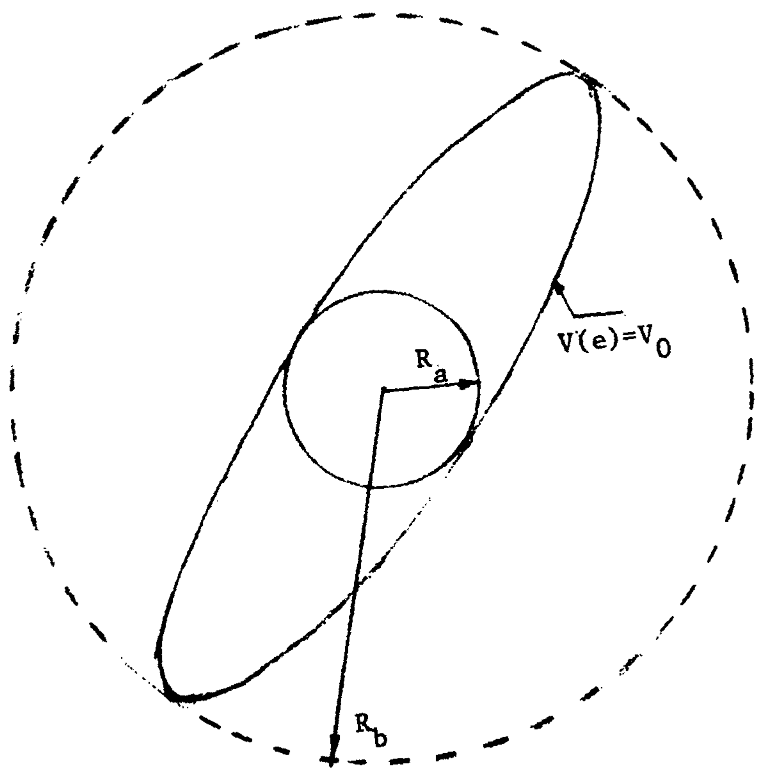
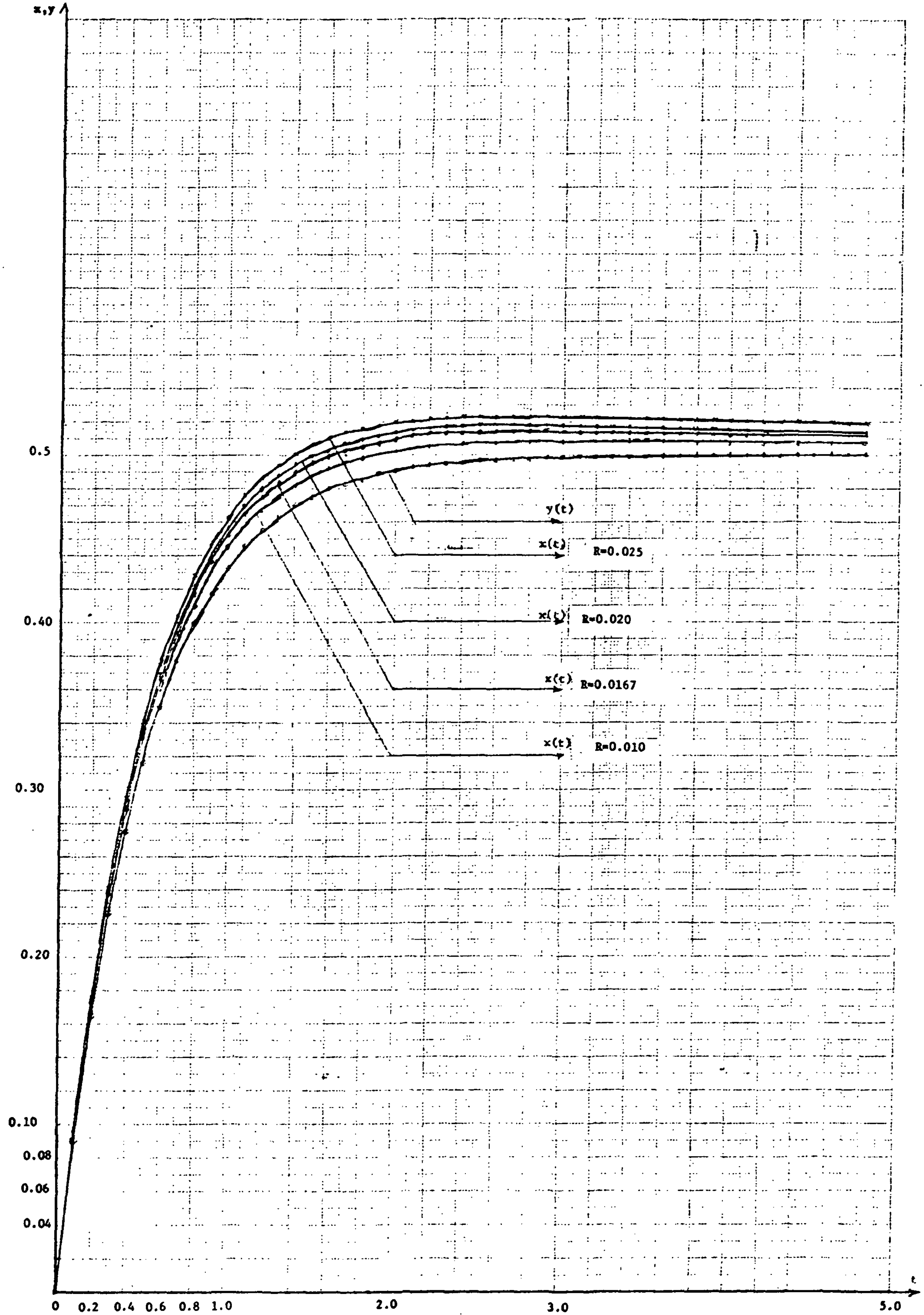


FIGURE 4



System Response for Different Values of R

Figure 5

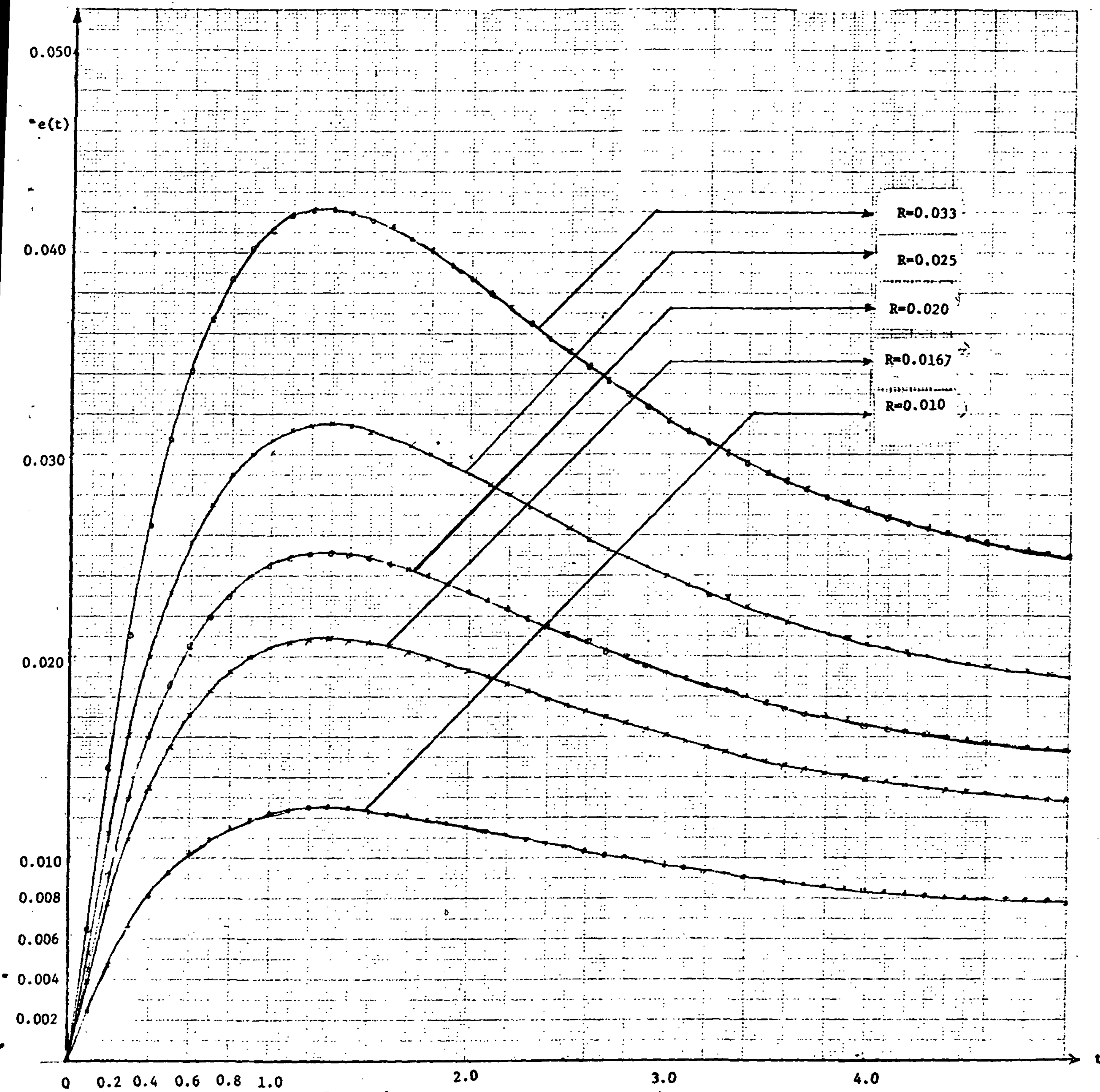


Figure 6 - Error between System output and model output for different values of R

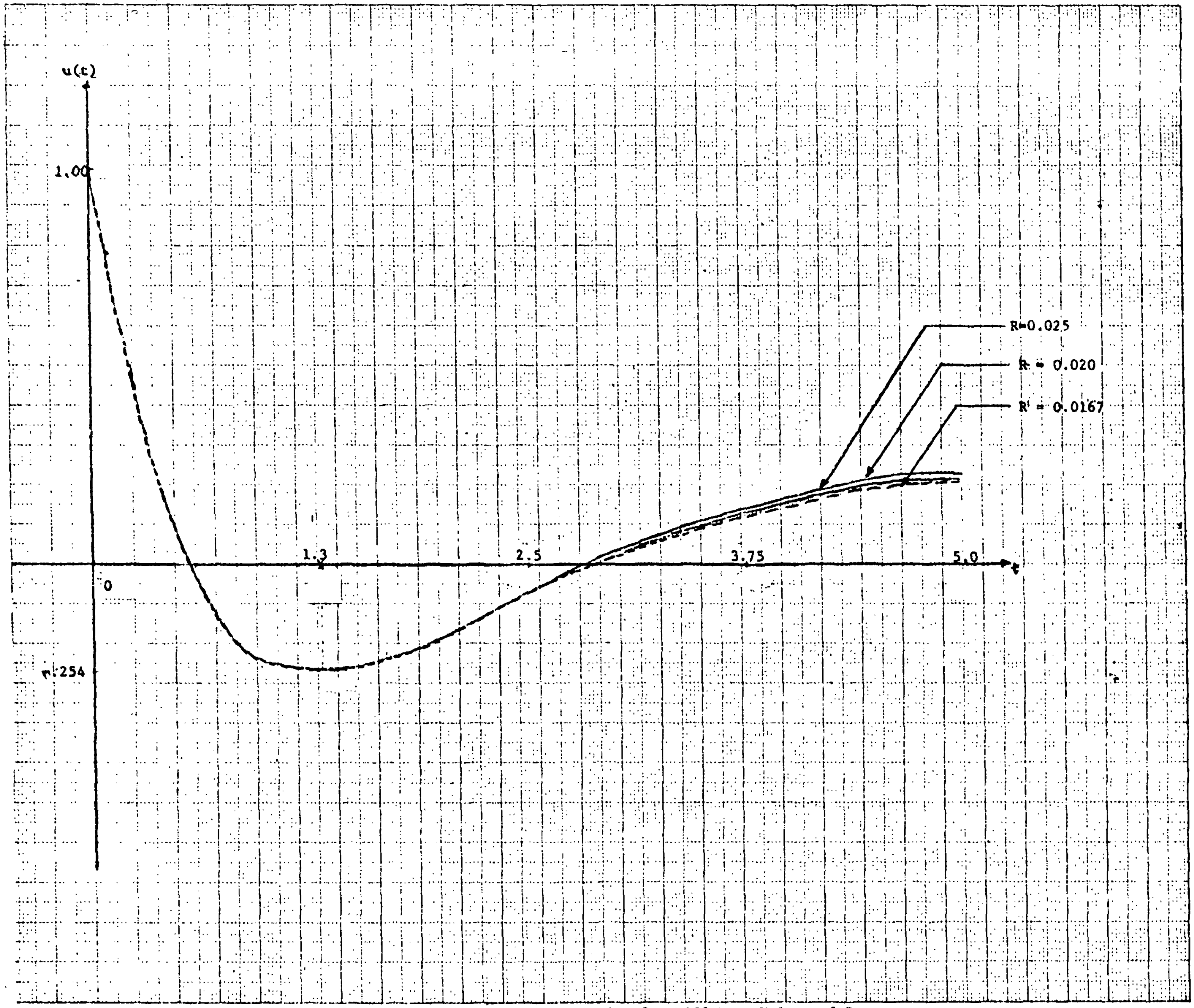


Figure 7 - Control Signal Characteristics for Different Values of R

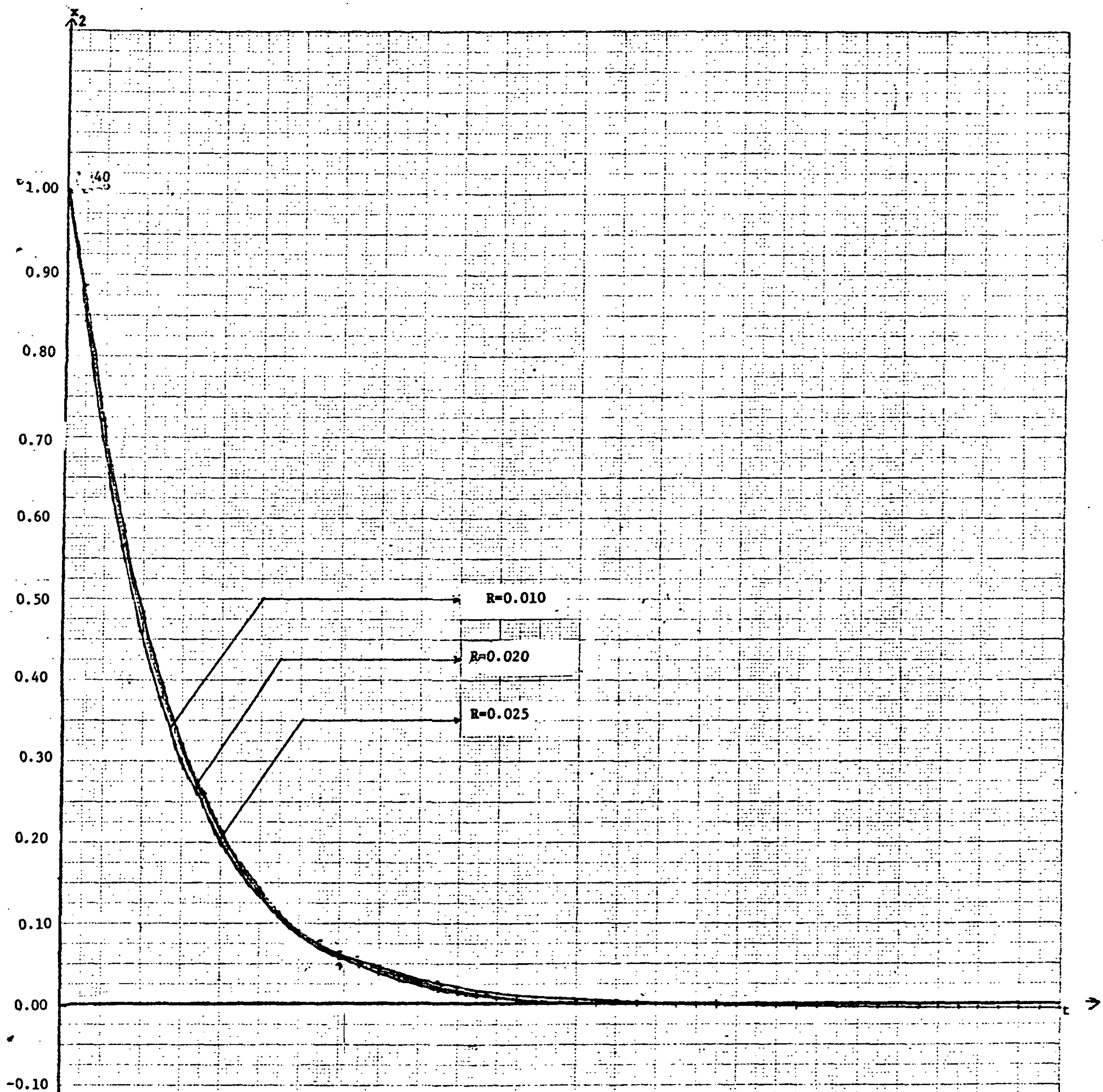


Figure 8 - Characteristics of Higher Order State x_2 of the System for Different Values of R

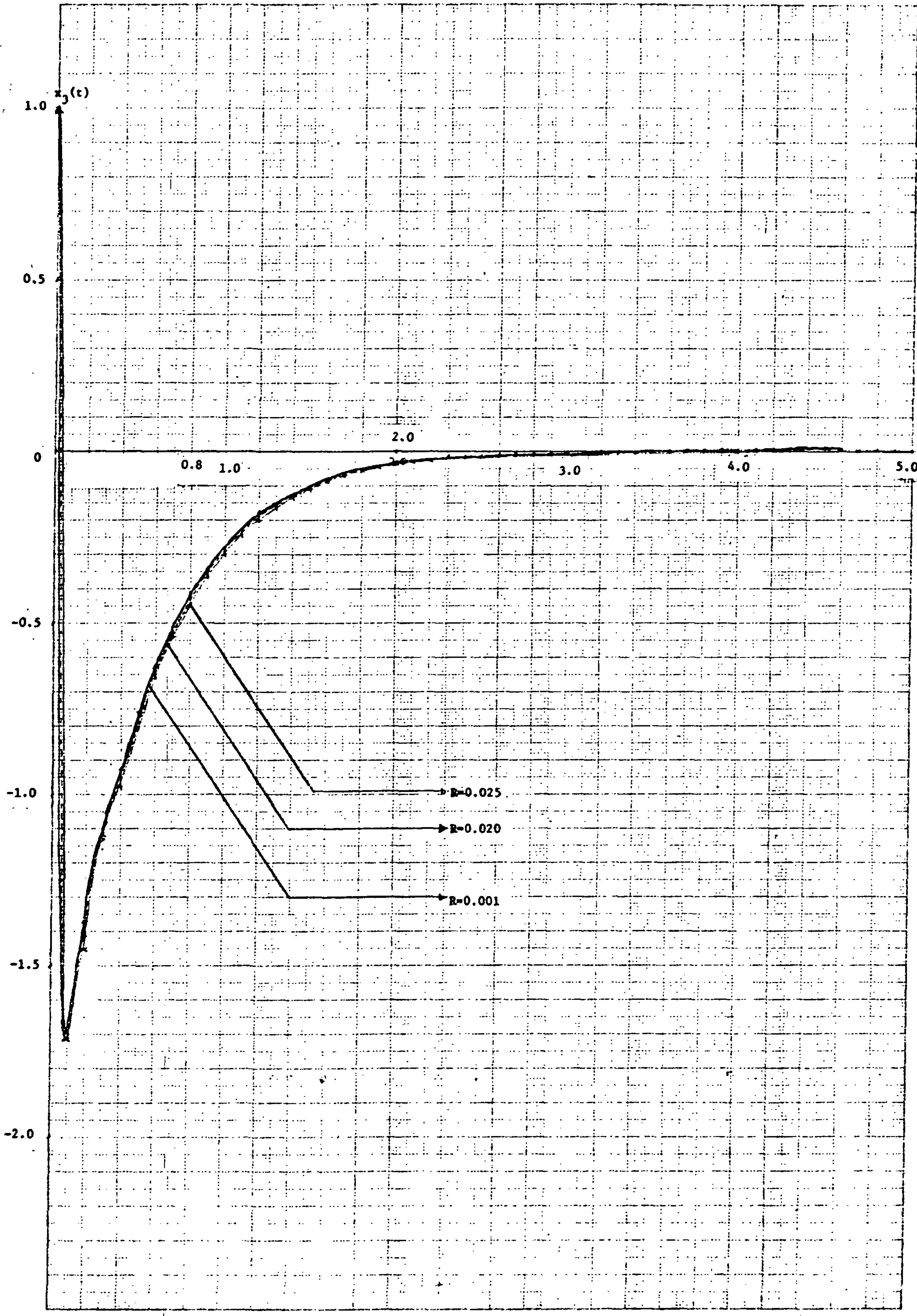


Figure 9 - Characteristics of higher Order State x_3 of the system for different values of R